SD

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Chapter 4: Laplace Transforms

Section 4-8: Dirac Delta Function

While Heaviside functions are useful when considering forcing functions that can be switched on and off, how would we deal with a forcing function that exerts a large force over a small interval of time (say a hammer striking an object)? The answer is the Dirac Delta function.

There are many ways to define the Dirac Delta function. Rather than focusing on its definition, we will make use of the following properties:

P 1. $\delta(t-a) = 0$ whenever $t \neq a$, in other words, $\delta(t-a) \cong \begin{cases} \infty, & \text{if } t=a \\ 0, & \text{if } t \neq 0 \end{cases}$

P 2. $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$, and

P 3. $\displaystyle\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a)$.

While the Dirac Delta function is not a real function as we think of them, it does a nice job of modeling sudden shocks or large forces to a system. Before moving on, we'll need to know how the Laplace transform acts on the Dirac Delta function.

Example 79. Use the integral definition of the Laplace transform to show that $\mathscr{L}\{\delta(t-a)\} = e^{-as}$ whenever $a > 0$.

$$\mathscr{L}\{\delta(t-a)\}(s) = \int_0^\infty e^{-st}\,\delta(t-a)\,dt, \quad (a>0)$$

$$(\text{by P1}) \quad = \int_{-\infty}^{\infty} e^{-st}\,\delta(t-a)\,dt$$

$$(\text{by P3}) \quad = e^{-sa} \qquad (a>0)$$

Example 80. Solve
$$y'' + 2y' + 2y = \delta(t-\pi), \quad y(0) = 1, \quad y'(0) = 0.$$

$$\mathcal{L}\{y'' + 2y' + 2y\} = \underbrace{\mathcal{L}\{\delta(t-\pi)\}}_{= e^{-\pi s}}$$

Let $Y(s) := \mathcal{L}\{y\}$

$$\Rightarrow \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = e^{-\pi s}$$

$$\Rightarrow \left(s^2 Y(s) - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0}\right) + 2\left(sY(s) - \underbrace{y(0)}_{=1}\right) + 2Y(s) = e^{-\pi s}$$

$$\Rightarrow \underline{s^2 Y(s)} - s + \underline{2sY(s)} - 2 + \underline{2Y(s)} = e^{-\pi s}$$

$$\Rightarrow (s^2 + 2s + 2)Y(s) - s - 2 = e^{-\pi s}$$

$$\Rightarrow (s^2 + 2s + 2)Y(s) = e^{-\pi s} + s + 2$$

$$\Rightarrow Y(s) = \frac{e^{-\pi s} + s + 2}{s^2 + 2s + 2}.$$

Notice $s^2 + 2s + 2 = s^2 + 2s + 1 + 1$
$= (s+1)^2 + 1$

$$\Rightarrow Y(s) = \frac{e^{-\pi s}}{(s+1)^2 + 1} + \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} \quad (*)$$

We will use following shifting properties to take the inverse Laplace Transform of $(*)$:

S1. $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

S2. $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)$

$$y(t) = \mathcal{L}^{-1}\{Y\}(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+1}\right\}}_{\mathrm{I}} + \underbrace{\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\}}_{\mathrm{II}} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}}_{\mathrm{III}}$$

$\underline{\mathrm{III}}$: By S1, $e^{at}f(t) = \underbrace{\mathcal{L}^{-1}\{F(s-a)\}}_{\substack{\text{comparing this to } \mathrm{III} \\ a=-1}}$

where $F(s) = \dfrac{1}{s^2+1}$, so $f(t) = \mathcal{L}^{-1}\{F\}(t) = \sin(t)$

Then, by S1, $\underline{\mathrm{III}} = \mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2+1}\right\} = e^{-t}\sin(t)$

Similarly, by S1, $F(s) = \dfrac{s}{s^2+1}$, so $f(t) = \cos(t)$,

hence $\underline{\mathrm{II}} = \mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+1}\right\} = e^{-t}\cos(t)$

I : Similar to III : $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2+1}\right\} = e^{-t}\sin(t) = f(t)$

By S2,

$$u_c(t)\, f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\} = \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{(s+1)^2+1}\right\}$$

$\parallel$

$$u_\pi(t)\cdot e^{-(t-\pi)}\sin(t-\pi) = \mathrm{I}$$

Thus,

$$y(t) = \mathrm{I} + \underline{\mathrm{II}} + \underline{\mathrm{III}} = u_\pi(t)\, e^{\pi-t}\sin(t-\pi) + e^{-t}\cos(t) + e^{-t}\sin(t)$$